Exercise 1: Constraint systems, Redundancy Elimination

Consider the following function.

```
int a, b, c;
c = 20;
a = 2*c + 10;
for (int i = 0; i < n; i++) {
    b = 2*c + 10;
    a += b / c + 42;
}
```

a) Draw the corresponding Control Flow Graph.
b) Using the approach from slide 46, write down the constraint system for calculating available expressions before the memorization transformation.
c) Compute the best solution of the constraint system using the fix-point computation approach from slide 101.
d) Perform the transformations for memorization and simple redundancy elimination for all non-trivial assignments in your graph.

Solution:

- CFG: see Figure 1.(a).
- Constraints (ignoring trivial expressions):

\[
\begin{align*}
\mathcal{A}[1] &\subseteq \emptyset \\
\mathcal{A}[2] &\subseteq (\mathcal{A}[1] \cup \emptyset) \setminus \text{Expr}_c \\
\mathcal{A}[3] &\subseteq (\mathcal{A}[2] \cup \{2 \times c + 10\}) \setminus \text{Expr}_a \\
\mathcal{A}[4] &\subseteq (\mathcal{A}[3] \cup \emptyset) \setminus \text{Expr}_i \\
\mathcal{A}[4] &\subseteq (\mathcal{A}[7] \cup \{i + 1\}) \setminus \text{Expr}_i \\
\mathcal{A}[5] &\subseteq \mathcal{A}[4] \cup \{i < n\} \\
\mathcal{A}[6] &\subseteq (\mathcal{A}[5] \cup \{2 \times c + 10\}) \setminus \text{Expr}_b \\
\mathcal{A}[7] &\subseteq (\mathcal{A}[6] \cup \{a + b/c + 42\}) \setminus \text{Expr}_a \\
\mathcal{A}[8] &\subseteq \mathcal{A}[4] \cup \{i < n\}
\end{align*}
\]
where \( \text{Expr}_a = \{a+b/c+42\} \), \( \text{Expr}_b = \{a+b/c+42\} \), \( \text{Expr}_c = \{2*c+10, a+b/c+42\} \), \( \text{Expr}_i = \{i+1, i < n\} \).

- We rewrite the constraint system to have one inequality per unknown. Thus we replace the two constraints for \( \mathcal{A}[4] \) by:

\[
\mathcal{A}[4] \subseteq (\mathcal{A}[3] \setminus \text{Expr}_i) \cap ((\mathcal{A}[7] \cup \{i + 1\}) \setminus \text{Expr}_i) = (\mathcal{A}[3] \cap (\mathcal{A}[7] \cup \{i + 1\})) \setminus \text{Expr}_i
\]

Also, for readability, we let \( e_1 = 2 * c + 10 \), \( e_2 = i < n \), \( e_3 = a + b/c + 42 \), \( e_4 = i + 1 \), and we write \( \bar{\mathcal{A}} \) for the sequence \( (\mathcal{A}[1], \ldots, \mathcal{A}[8]) \) of unknowns.

Figure 1: CFGs for 1.(a) and 1.(d)
The function \( F \) that we iterate is thus given by \( F(\bar{A}) = (f_1(\bar{A}), \ldots, f_8(\bar{A})) \), where the functions \( f_i \) are as follows:

\[
\begin{align*}
  f_1(\bar{A}) &= \emptyset \\
  f_2(\bar{A}) &= \mathcal{A}[1] \setminus \{e_1, e_3\} \\
  f_3(\bar{A}) &= (\mathcal{A}[2] \cup \{e_1\}) \setminus \{e_3\} \\
  f_4(\bar{A}) &= (\mathcal{A}[3] \cap (\mathcal{A}[7] \cup \{e_4\}) \setminus \{e_2, e_4\} \\
  f_5(\bar{A}) &= \mathcal{A}[4] \cup \{e_2\} \\
  f_6(\bar{A}) &= (\mathcal{A}[5] \cup \{e_1\}) \setminus \{e_3\} \\
  f_7(\bar{A}) &= \mathcal{A}[6] \setminus \{e_3\} \\
  f_8(\bar{A}) &= \mathcal{A}[4] \cup \{e_2\}
\end{align*}
\]

The following table gives the \( i \)th component of \( F^j(\bar{A}) \), for each component \( i \in \{1, \ldots, 8\} \) and each iteration \( j \).

<table>
<thead>
<tr>
<th>( i \backslash j )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{e_1, e_2, e_3, e_4}</td>
<td>\emptyset</td>
<td>\emptyset</td>
<td>\emptyset</td>
<td>\emptyset</td>
<td>\emptyset</td>
</tr>
<tr>
<td>2</td>
<td>{e_1, e_2, e_3, e_4}</td>
<td>{e_2, e_4}</td>
<td>\emptyset</td>
<td>\emptyset</td>
<td>\emptyset</td>
<td>\emptyset</td>
</tr>
<tr>
<td>3</td>
<td>{e_1, e_2, e_3, e_4}</td>
<td>{e_2, e_4}</td>
<td>{e_1, e_2, e_4}</td>
<td>{e_1}</td>
<td>{e_1}</td>
<td>{e_1}</td>
</tr>
<tr>
<td>4</td>
<td>{e_1, e_2, e_3, e_4}</td>
<td>{e_1}</td>
<td>{e_1, e_3}</td>
<td>{e_1}</td>
<td>{e_1}</td>
<td>{e_1}</td>
</tr>
<tr>
<td>5</td>
<td>{e_1, e_2, e_3, e_4}</td>
<td>{e_1, e_2, e_3}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
</tr>
<tr>
<td>6</td>
<td>{e_1, e_2, e_3, e_4}</td>
<td>{e_1, e_2, e_4}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
</tr>
<tr>
<td>7</td>
<td>{e_1, e_2, e_3, e_4}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2, e_4}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
</tr>
<tr>
<td>8</td>
<td>{e_1, e_2, e_3, e_4}</td>
<td>{e_1, e_2, e_3}</td>
<td>{e_1, e_2, e_4}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
<td>{e_1, e_2}</td>
</tr>
</tbody>
</table>

- After transformations, we obtain the CFG in Figure 1.(b).

**Exercise 2: Least fixpoint**

For a lattice over natural numbers \( \mathbb{N} \) with ordering \( \leq \), give the least fixpoint for the following function.

\[
f(x) = \min(x + 1, 100)
\]

Prove that the given number is the least fixpoint.

**Solution:** Least fixpoint of \( f \) is 100, as

- it is a fixpoint, e.g. \( f(100) = \min(101, 100) = 100 \), and
- it is less than any other fixpoint, e.g. if \( f(p) = p \) then \( p \geq 100 \), because for \( 0 \leq q < 100 \), \( f(q) = q + 1 \) and obviously \( q \neq q + 1 \) is not a fixpoint.

**Exercise 3: Lattice Theory**

For a lattice \( \mathbb{D} = (D, \sqsubseteq) \), we define the height of the lattice \( h(D) = n \) as the maximal length of strictly ascending chains \( d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \) in the lattice with \( d_i \in D \).

1. Finiteness of height does not imply finiteness of a lattice. Give an example lattice!
2. Let $\mathbb{D}_1 = (D_1, \sqsubseteq_1)$ and $\mathbb{D}_2 = (D_2, \sqsubseteq_2)$ be lattices of finite height. We define a product lattice $\mathbb{D}_1 \times \mathbb{D}_2$ as the lattice $(D_1 \times D_2, \sqsubseteq)$ with $(a, b) \sqsubseteq (a', b')$ iff $a \sqsubseteq_1 a'$ and $b \sqsubseteq_2 b'$ for $a, a' \in D_1$ and $b, b' \in D_2$.

Show that $h(\mathbb{D}_1 \times \mathbb{D}_2) = h(\mathbb{D}_1) + h(\mathbb{D}_2)$.

Solution:

1. An example lattice is $\mathbb{Z}_+^\top = \mathbb{Z} \cup \{\bot, \top\}$ with the relation $\bot \sqsubseteq z$ and $z \sqsubseteq \top$ and $z \sqsubseteq z$ with $z \in \mathbb{Z}_+^\top$. The height is obviously two but the number of elements is infinite.

2. For, $h(\mathbb{D}_1 \times \mathbb{D}_2) = h(\mathbb{D}_1) + h(\mathbb{D}_2)$, we abbreviate $k, n$ and $m$, for the heights of $\mathbb{D}_1 \times \mathbb{D}_2$, $\mathbb{D}_1$, and $\mathbb{D}_2$, respectively. We prove that $k = n + m$ by showing first that $k \geq n + m$, and second that $k \leq n + m$.

Case $k \geq n + m$: We know a sequence $a_0 \sqsubseteq_1 a_1 \sqsubseteq_1 \cdots \sqsubseteq_1 a_n$ exists in $\mathbb{D}_1$ and a sequence $b_0 \sqsubseteq_2 b_1 \sqsubseteq_2 \cdots \sqsubseteq_2 b_m$ exists in $\mathbb{D}_2$. We form the strictly ascending chain

$$(a_0, b_0) \sqsubseteq (a_1, b_0) \sqsubseteq \cdots \sqsubseteq (a_n, b_0) \sqsubseteq (a_0, b_1) \sqsubseteq \cdots \sqsubseteq (a_n, b_m)$$

in $\mathbb{D}_1 \times \mathbb{D}_2$. It has length $n + m$. It follows that i.e., $k \geq n + m$.

Case $k \leq n + m$: We know that a strictly ascending chain $c_0 \sqsubseteq c_1 \sqsubseteq \cdots \sqsubseteq c_k$ exists in $\mathbb{D}_1 \times \mathbb{D}_2$. Let $c_i = (a_i, b_i)$, for $i$ between 0 and $k$. For each $i$ between 0 and $k - 1$, we have that $a_i \sqsubseteq_1 a_{i+1}$ and $b_i \sqsubseteq_2 b_{i+1}$, and at least one of the inequalities is strict (as otherwise $c_i = c_{i+1}$). Let $A = \{i \mid 0 \leq a_i \sqsubseteq_1 a_{i+1}\}$ and $B = \{j \mid 0 \leq b_j \sqsubseteq_2 b_{j+1}\}$. The elements $a_i$ with $i \in A$ and elements $b_j$ with $j \in B$ form strictly ascending chains in $\mathbb{D}_1$ and $\mathbb{D}_2$ respectively. Thus $|A| \leq n$ and $|B| \leq m$. We have that $|A \cup B| \leq |A| + |B| \leq n + m$. We also have that $|A \cup B| \geq k$, because each index $i$ between 0 and $k - 1$ is in $A \cup B$ (otherwise $a_i = a_{i+1}$ and $b_i = b_{i+1}$ and thus $c_i = c_{i+1}$—a contradiction). Thus $k \leq n + m$.

We conclude that $k = n + m$.

Exercise 4: Lattice Theory

A lower semilattice $(\mathbb{D}, \sqsubseteq)$ is a partial order such that each two elements $a, b \in \mathbb{D}$ have a greatest lower bound, written $a \sqcap b$ (also called meet). An upper semilattice $(\mathbb{D}, \sqsubseteq)$ is a partial order such that each two elements $a, b \in \mathbb{D}$ have a lowest upper bound, written $a \sqcup b$ (also called join). A lattice is a partial order that is both a lower and an upper semilattice.

Which of the following are a partial order, a lower/upper semilattice, or a complete lattice? (Give a short justification for your answers!)

1.
2. Strings over an alphabet $\Sigma = \{0, 1\}$ with subsequence ordering, i.e., we have $x \sqsubseteq y$ if $x$ can be obtained by deleting letters from $y$. For instance, $\epsilon \sqsubseteq x$, for any $x \in \Sigma^*$; $1 \sqsubseteq 10; 0 \sqsubseteq 10; 11 \sqsubseteq 010010$.

3. $(\mathbb{Z}, \leq)$.

*Solution:* We have Complete lattices $\subset$ Lattices $\subset$ Lower semilattices $\subset$ Partial orders and Complete lattices $\subset$ Lattices $\subset$ Upper semilattices $\subset$ Partial orders.

1. The upper bounds of $b$ and $c$ are $\{\top, h, i\}$. Since $h \not\sqsubseteq i$ and $i \not\sqsubseteq h$, no least upper bound exists for $b$ and $c$, hence it is not a upper semilattice.

The lower bounds of $e$ and $g$ are $\{\bot, a, c\}$. Since $a \not\sqsubseteq c$ and $c \not\sqsubseteq a$, no greatest lower bound exists for $e$ and $i$, hence it is also not a lower semilattice.

So it is only a partial order.

2. There is no string that every other string is a subsequence of. Since we do not have an upper bound, it is not a complete lattice.

It is also not a lower semilattice since there is no distinct meet (e.g. for 01 and 10 we have $0 \sqsubseteq 10, 0 \sqsubseteq 10$, and also $1 \sqsubseteq 10, 1 \sqsubseteq 10; 0$ and 1 are not comparable).

It is also not an upper semilattice since there is no distinct join (e.g. for 01 and 10 we have $01 \sqsubseteq 101, 10 \sqsubseteq 101$, and also $01 \sqsubseteq 010, 10 \sqsubseteq 010; 101$ and 010 are not comparable).

So it is only a partial order.

3. Since it is not bounded, it is not a complete lattice. However, for any two elements, we have $a \sqcap b = \min(a, b)$ and $a \sqcup b = \max(a, b)$, so it is a lattice.