Exercise 4.1
Using the Compare feature in Spot (https://spot.lrde.epita.fr/app) give an LTL formula equivalent to
(a) $p R q$, which does not contain $\neg$ but may contain $U, G$ or $F$.
(b) $(Gp) U q$ which does not contain $U$.
(c) $(Fp) U q$, which does not contain $U$.

Exercise 4.2
Think of a way to use Spot to check if a word $\alpha$ satisfies an LTL formula $\phi$. Check if the word $\{q\}\{s\}\{p\}^{\omega}$ satisfies $G\neg q \lor F(q \land \neg p W s)$.

Exercise 4.3
Challenge: what is the largest LTL formula you can come up with using only one atomic proposition $p$ and without using the $X$ operator, which Spot is unable to simplify?

Exercise 4.4
Given the following Kripke structures and LTL formulae, answer the following questions
(a) Which of $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ satisfy $\phi = G(Xq \rightarrow p)$?
(b) Give an LTL formula which exactly characterizes $\mathcal{K}_3$, i.e. both the formula and the Kripke structure accept exactly the same words.

Exercise 4.5
Let $AP = \{p, q\}$ and let $\Sigma = 2^{AP}$. Give Büchi automata recognizing the $\omega$-languages over $\Sigma$ defined by the following LTL formulas:
(a) $XG\neg p$
(b) $(GFp) \rightarrow (Fq)$
(c) $p \land \neg(XFp)$
(d) $G(p U (p \rightarrow q))$
(e) $Fq \rightarrow (\neg q U (\neg q \land p))$

Exercise 4.6
Given $L = \{\{p\}^m \{q\}^n\}^\omega : m \leq n$, show that there is no Büchi automata recognizing $L$. 
Solution 4.1
(a) \((q \lor (p \land q)) \lor Gq\)

(b) \((Fq \land Gp) \lor q\).

(c) \(q \lor F(Fp \land Xq)\) or another longer solution: \(F(q \land Fp) \lor F(p \land Xq) \lor Gq \lor q\)

Solution 4.2
Use \(X, XX\) and so on to describe the word. Then run compare.

Solution 4.3
\[((p \land F \land G(Fp \land F(G \land Fp \land F(p \land Fp))) \land (F(G \land Fp \land F(G \land Fp))) \land ((p \land F(G \land Fp \land F(G \land Fp))) \land (p \land F(G \land Fp \land F(G \land Fp))) \land ((p \land F(G \land Fp \land F(G \land Fp))) \land (p \land F(G \land Fp \land F(G \land Fp))) \land ((p \land F(G \land Fp \land F(G \land Fp))) \land (p \land F(G \land Fp \land F(G \land Fp))))\)
Solution 4.4
(a) $\mathcal{K}_1$
(b) $G\neg p \lor (\neg p \mathbf{U} G(p \rightarrow Xq \land \neg p \rightarrow Xp))$

Solution 4.5
(a)
(b) Note that $(GFp) \rightarrow (Fq) \equiv \neg (GFp) \lor (Fq) \equiv (FG\neg p) \lor (Fq)$. We construct Büchi automata for $FG\neg p$ and $Fq$, and take their union:
(c) Note that $p \land \neg (XFp) \equiv p \land XG \neg p$. We construct a Büchi automaton for $p \land XG \neg p$:

\[
\begin{array}{c}
\{p\}. \{p, q\} \\
\{p\}
\end{array}
\]

(d)

\[
\begin{array}{c}
\{p\} \\
\emptyset, \{q\}, \{p, q\} \\
\emptyset, \{q\}, \{p, q\} \\
\{p\}
\end{array}
\]

(e)

\[
\begin{array}{c}
\emptyset \\
\{p\} \\
\Sigma
\end{array}
\]

Solution 4.6

For the sake of contradiction, suppose there exists a Büchi automaton $B = (Q, \Sigma, \delta, Q_0, F)$ such that $L(B) = L$. Let $m = |Q|$ and let $\sigma = \{p\}^m \{q\}^m \emptyset^\omega$. Since $\sigma \in L(B)$, there exist $q_0, q_1, \ldots \in Q$ such that $q_0 \in Q_0$, there are infinitely many indices $i$ such that $q_i \in F$ and

\[
q_0 \xrightarrow{\sigma_0} q_1 \xrightarrow{\sigma_1} q_2 \cdots .
\]

By the pigeonhole principle, there exist $0 \leq i < j \leq m$ such that $q_i = q_j$. Let $u = \sigma_0 \sigma_1 \cdots \sigma_{i-1}$, $v = \sigma_i \sigma_{i+1} \cdots \sigma_{j-1}$ and $w = \sigma_j \sigma_{j+1} \cdots$. We have:

\[
q_0 \xrightarrow{u} q_i \xrightarrow{v^{m+1}} q_j \xrightarrow{w} \cdots
\]

Thus, $\sigma' \in L(B)$ where $\sigma' = wv^{m+1}w$. Note that $v$ solely consists of the letter $\{p\}$, hence $|P_{\sigma'}| \geq m + 1 > m = |Q_{\sigma'}|$, which contradicts $\sigma \in L(B) = L$. 

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