Fundamental Algorithms

Chapter 2: Sorting

Jan Křetínský

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Part I

Simple Sorts
The Sorting Problem

Definition

Sorting is required to order a given sequence of elements, or more precisely:

**Input** : a sequence of \( n \) elements \( a_1, a_2, \ldots, a_n \)

**Output** : a permutation (reordering) \( a'_1, a'_2, \ldots, a'_n \) of the input sequence, such that \( a'_1 \leq a'_2 \leq \cdots \leq a'_n \).

- we will assume the elements \( a_1, a_2, \ldots, a_n \) to be integers (or any element/data type on which a total order \( \leq \) is defined)
- a sorting algorithm may output the permuted data or also the permuted set of indices
Insertion Sort

Idea: sorting by inserting

- successively generate ordered sequences of the first \( j \) numbers: \( j = 1, j = 2, \ldots, j = n \)
- in each step, \( j \rightarrow j + 1 \), one additional integer has to be inserted into an already ordered ordered sequence

Data Structures:

- an array \( A[1..n] \) that contains the sequence \( a_1 \) (in \( A[1] \)), \( \ldots \), \( a_n \) (in \( A[n] \)).
- numbers are sorted in place: output sequence will be stored in \( A \) itself (hence, content of \( A \) is changed)
Insertion Sort – Implementation

InsertionSort(A: Array[1..n]) {

    for j from 2 to n {
        // insert A[j] into sequence A[1..j−1]

        key := A[j];

        i := j−1; // initialize i for while loop
        while i >= 1 and A[i] > key {
            A[i+1] := A[i];
            i := i−1;
        }
        A[i+1] := key;
    }
}
Correctness of InsertionSort

Loop invariant:

Before each iteration of the for-loop, the subarray $A[1..j-1]$ consists of all elements originally in $A[1..j-1]$, but in sorted order.

Initialization:

- loops starts with $j=2$; hence, $A[1..j-1]$ consists of the element $A[1]$ only
Correctness of InsertionSort

Loop invariant:

Before each iteration of the for-loop, the subarray A[1..j-1] consists of all elements originally in A[1..j-1], but in sorted order.

Maintenance:

• assume that the while loop works correctly (or prove this using an additional loop invariant):
  • after the while loop, i contains the largest index for which A[i] is smaller than the key
  • A[i+2..j] contains the (sorted) elements previously stored in A[i+1..j-1]; also: A[i+1] and all elements in A[i+2..j] are ≥ key
• the key value, A[j], is thus correctly inserted as element A[i+1] (overwrites the duplicate value A[i+1])
• after execution of the loop body, A[1..j] is sorted
• thus, before the next iteration (j:=j+1), A[1..j-1] is sorted
Correctness of InsertionSort

Loop invariant:

Before each iteration of the for-loop, the subarray $A[1..j-1]$ consists of all elements originally in $A[1..j-1]$, but in sorted order.

Termination:

- The for-loop terminates when $j$ exceeds $n$ (i.e., $j=n+1$)
- Thus, at termination, $A[1 .. (n+1)-1] = A[1..n]$ is sorted and contains all original elements
Insertion Sort – Number of Comparisons

InsertionSort \((A: \text{Array}[1..n])\) \{

\[
\text{for } j \text{ from } 2 \text{ to } n \{ \\
\text{key := A}[j]; \\
\text{i := j} - 1; \\
\text{while } i >= 1 \text{ and } A[i] > \text{key} \{ \\
\text{A}[i+1] := A[i]; \\
\text{i := i} - 1; \\
\} \\
\text{A}[i+1] := \text{key;}
\}
\]

\(n - 1\) iterations

\[t_j\) iterations \Rightarrow \sum_{j=2}^{n} t_j\) comparisons

\(t_j\) comparisons

\(A[i] > \text{key}\)
Insertion Sort – Number of Comparisons (2)

- counted number of comparisons: \( T_{\text{IS}} = \sum_{j=2}^{n} t_j \)
- where \( t_j \) is the number of iterations of the while loop (which is, of course, unknown)
- good estimate for the run time, if the comparison is the most expensive operation (note: replace “\( i \geq 1 \)” by for loop)

Analysis
- what is the “best case”?
- what is the “worst case”? 
Insertion Sort – Number of Comparisons (2)

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• good estimate for the run time, if the comparison is the most expensive operation (note: replace “\( i \geq 1 \)” by for loop)

Analysis of the “best case”:

• in the best case, \( t_j = 1 \) for all \( j \)

• happens only, if \( A[1..n] \) is already sorted

\[ T_{IS}(n) = \sum_{j=2}^{n} 1 = n - 1 \in \Theta(n) \]
Insertion Sort – Number of Comparisons (2)

• counted number of comparisons: \( T_{IS} = \sum_{j=2}^{n} t_j \)

• where \( t_j \) is the number of iterations of the while loop (which is, of course, unknown)

• good estimate for the run time, if the comparison is the most expensive operation (note: replace “\( i \geq 1 \)” by for loop)

Analysis of the “worst case”:
• in the worst case, \( t_j = j - 1 \) for all \( j \)
• happens, if \( A[1..n] \) is already sorted in opposite order

\[
\Rightarrow T_{IS}(n) = \sum_{j=2}^{n} (j - 1) = \frac{1}{2} n(n - 1) \in \Theta(n^2)
\]
Insertion Sort – Number of Comparisons (2)

• counted number of comparisons: \( T_{IS} = \sum_{j=2}^{n} t_j \)

• where \( t_j \) is the number of iterations of the while loop (which is, of course, unknown)

• good estimate for the run time, if the comparison is the most expensive operation (note: replace “i > =1” by for loop)

Analysis of the “average case”:

• best case analysis: \( T_{IS}(n) \in \Theta(n) \)

• worst case analysis: \( T_{IS}(n) \in \Theta(n^2) \)

⇒ What will be the ”typical” (average, expected) case?
Running Time and Complexity

“Run(ning )Time”

- the notation $T(n)$ suggest a “time”, such as run(ning) time of an algorithm, which depends on the input (size) $n$
- in practice: we need a precise model how long each operation of our programmes takes $\rightarrow$ very difficult on real hardware!
- we will therefore determine the number of operations that determine the run time, such as:
  - number of comparisons (sorting, e.g.)
  - number of arithmetic operations (Fibonacci, e.g.)
  - number of memory accesses
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“Complexity”

- characterises how the run time depends on the input (size), typically expressed in terms of the $\Theta$-notation
- “algorithm xyz has linear complexity” $\rightarrow$ run time is $\Theta(n)$
Average Case Complexity

**Definition (expected running time)**

Let \( X(n) \) be the set of all possible input sequences of length \( n \), and let \( P: X(n) \to [0, 1] \) be a probability function such that \( P(x) \) is the probability that the input sequence is \( x \).

Then, we define

\[
\bar{T}(n) = \sum_{x \in X(n)} P(x)T(x)
\]

as the *expected running time* of the algorithm.

**Comments:**

- we require an exact probability distribution
  (for InsertionSort, we could assume that all possible sequences have the same probability)

- we need to be able to determine \( T(x) \) for any sequence \( x \)
  (usually much too laborious to determine)
Average Case Complexity of Insertion Sort

Heuristic estimate:
- we assume that we need $\frac{j}{2}$ steps in every iteration:

$$\Rightarrow \bar{T}_{IS}(n) \approx \sum_{j=2}^{n} \frac{j}{2} = \frac{1}{2} \sum_{j=2}^{n} j \in \Theta(n^2)$$
Average Case Complexity of Insertion Sort

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- note: $\frac{j}{2}$ isn’t even an integer . . .
Average Case Complexity of Insertion Sort

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- note: $\frac{j}{2}$ isn’t even an integer . . .
- Just considering the number of comparisons of the “average case” can lead to quite wrong results!

in general $E(T(n)) \neq T(“E(n)”)$
Bubble Sort

BubbleSort(A: Array[1..n]) {
    for i from 1 to n do {
        for j from n downto i+1 do {
        }
    }
}

Basic ideas:

• compare neighboring elements only
• exchange values if they are not in sorted order
• repeat until array is sorted (here: pessimistic loop choice)
Bubble Sort – Homework

Prove correctness of Bubble Sort:

- find invariant for i-loop
- find invariant for j-loop

Number of comparisons in Bubble Sort:

- best/worst/average case?
Part II

Mergesort and Quicksort
Mergesort

Basic Idea: divide and conquer

- **Divide** the problem into two (or more) subproblems:
  → split the array into two arrays of equal size
- **Conquer** the subproblems by solving them recursively:
  → sort both arrays using the sorting algorithm
- **Combine** the solutions of the subproblems:
  → merge the two sorted arrays to produce the entire sorted array
Combining Two Sorted Arrays: Merge

Merge (L: Array[1..p], R: Array[1..q], A: Array[1..n]) {
    // merge the sorted arrays L and R into A (sorted)
    // we presume that n=p+q
    i := 1; j := 1:
    for k from 1 to n do {
        if i > p
            then { A[k] := R[j]; j := j + 1; } 
        else if j > q
            then { A[k] := L[i]; i := i + 1; } 
        else if L[i] < R[j]
            then { A[k] := L[i]; i := i + 1; } 
        else { A[k] := R[j]; j := j + 1; } 
    }
}
Correctness and Run Time of Merge

Loop invariant:
Before each cycle of the for loop:
- A has the k-1 smallest elements of L and R already merged, (i.e. in sorted order and at indices 1, \ldots, k-1);
- L[i] and R[j] are the smallest elements of L and R that have not been copied to A yet (i.e. L[1..i-1] and R[1..j-1] have been merged to A)

Run time:
\[ T_{\text{Merge}}(n) \in \Theta(n) \]
- for loop will be executed exactly \( n \) times
- each loop contains constant number of commands:
  - exactly 1 copy statement
  - exactly 1 increment statement
  - 1–3 comparisons
MergeSort

```plaintext
MergeSort(A: Array[1..n]) {
    if n > 1 then {
        m := floor(n/2);
        create array L[1..m];
        for i from 1 to m do { L[i] := A[i]; }

        create array R[1..n-m];
        for i from 1 to n-m do { R[i] := A[m+i]; }

        MergeSort(L);
        MergeSort(R);

        Merge(L,R,A);
    }
}
```
Number of Comparisons in MergeSort

- Merge performs exactly \( n \) element copies on \( n \) elements
- Merge performs at most \( c \cdot n \) comparisons on \( n \) elements
- MergeSort itself does not contain any comparisons between elements; all comparisons done in Merge

\[ C_{\text{MS}}(n) = \begin{cases} 
0 & \text{if } n \leq 1 \\
C_{\text{MS}} \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + C_{\text{MS}} \left( n - \left\lfloor \frac{n}{2} \right\rfloor \right) + 2n & \text{if } n \geq 2
\end{cases} \]

⇒ number of comparisons for the entire MergeSort algorithm:

\[ T_{\text{MS}}(n) \leq \begin{cases} 
0 & \text{if } n \leq 1 \\
T_{\text{MS}} \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + T_{\text{MS}} \left( n - \left\lfloor \frac{n}{2} \right\rfloor \right) + cn & \text{if } n \geq 2
\end{cases} \]
Number of Comparisons in MergeSort (2)

Assume $n = 2^k$, $c$ constant:

$$T_{MS}(2^k) \leq T_{MS}(2^{k-1}) + T_{MS}(2^{k-1}) + c \cdot 2^k$$

$$\leq 2T_{MS}(2^{k-1}) + 2^k c$$
Number of Comparisons in MergeSort (2)

Assume \( n = 2^k \), \( c \) constant:

\[
T_{MS}(2^k) \leq T_{MS}(2^{k-1}) + T_{MS}(2^{k-1}) + c \cdot 2^k
\]

\[
\leq 2T_{MS}(2^{k-1}) + 2^k c
\]

\[
\leq 2^2 T_{MS}(2^{k-2}) + 2 \cdot 2^{k-1} c + 2^k c
\]

\[
\leq \ldots
\]

\[
\leq 2^k T_{MS}(2^0) + 2^k c + 2^k c + \ldots
\]

\[
\leq k \sum_{j=1}^k 2^{k-j} c + k \cdot 2^k c
\]

\[
\leq k \cdot 2^k c + k \cdot 2^k c
\]

\[
= 2^k c \cdot (k + 1)
\]

\[
\in O(n \log n)
\]
Number of Comparisons in MergeSort (2)

Assume $n = 2^k$, $c$ constant:

$$T_{MS}(2^k) \leq T_{MS}(2^{k-1}) + T_{MS}(2^{k-1}) + c \cdot 2^k$$

$$\leq 2T_{MS}(2^{k-1}) + 2^k c$$

$$\leq 2^2 T_{MS}(2^{k-2}) + 2 \cdot 2^{k-1} c + 2^k c$$

$$\leq \ldots$$

$$\leq 2^k T_{MS}(2^0) + 2^{k-1} \cdot 2^1 c + \ldots + 2^j \cdot 2^{k-j} c$$

$$\quad + \ldots + 2 \cdot 2^{k-1} c + 2^k c$$

$$\leq \sum_{j=1}^{k} 2^k c = ck \cdot 2^k = cn \log_2 n \in O(n \log n)$$
Quicksort

Basic Idea: divide and conquer

- **Divide** the input array $A[p..r]$ into parts $A[p..q]$ and $A[q+1..r]$, such that every element in $A[q+1..r]$ is larger than all elements in $A[p..q]$.
- **Conquer:** sort the two arrays $A[p..q]$ and $A[q+1..r]$
- **Combine:** if the divide and conquer steps are performed in place, then no further combination step is required.
Quicksort

**Basic Idea:** divide and conquer

- **Divide** the input array \(A[p..r]\) into parts \(A[p..q]\) and \(A[q+1 .. r]\), such that every element in \(A[q+1 .. r]\) is larger than all elements in \(A[p .. q]\).
- **Conquer:** sort the two arrays \(A[p..q]\) and \(A[q+1 .. r]\)
- **Combine:** if the divide and conquer steps are performed in place, then no further combination step is required.

Partitioning using a **pivot element**:

- all elements that are smaller than the pivot element should go into the “smaller” partition \((A[p..q])\)
- all elements that are larger than the pivot element should go into the “larger” partition \((A[q+1..r])\)
Partitioning the Array (Hoare’s Algorithm)

Partition (A: Array[p..r]) : Integer {
    // x is the pivot (chosen as first element):
    x := A[p];
    // partitions grow towards each other
    i := p−1; j := r+1; // (partition boundaries)
    while true do { // i<j: partitions haven’t met yet
        // leave large elements in right partition
        do { j:=j−1; } while A[j]>x;
        // leave small elements in left partition
        do { i:=i+1; } while A[i]<x;
        // swap the two first “wrong” elements
        if i < j then exchange A[i] and A[j];
        else return j;
    }
}
Time Complexity of Partition

How many statements are executed by the nested while loops?
How many statements are executed by the nested while loops?

- monitor increments/decrements of i and j
- after $n := r - p$ increments/decrements, i and j have the same value

$\Rightarrow \Theta(n)$ comparisons with the pivot
$\Rightarrow O(n)$ element exchanges

Hence: $T_{Part}(n) \in \Theta(n)$
Implementation of QuickSort

QuickSort (A:Array[p..r])
{
    if p>=r then return;
    // only proceed, if A has at least 2 elements:
    q := Partition (A);
    QuickSort (A[p..q]);
    QuickSort (A[q+1..r]);
}

Homework:

• prove correctness of Partition
• prove correctness of QuickSort
Time Complexity of QuickSort

Best Case:
• assume that all partitions are split exactly into two halves:

\[ T_{\text{QS}}^{\text{best}}(n) = 2 T_{\text{QS}}^{\text{best}} \left( \frac{n}{2} \right) + \Theta(n) \]

• analogous to MergeSort:

\[ T_{\text{QS}}^{\text{best}}(n) \in \Theta(n \log n) \]
Time Complexity of QuickSort

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- analogous to MergeSort:

\[ T_{QS}^{\text{best}}(n) \in \Theta(n \log n) \]

Worst Case:
- Partition will always produce one partition with only 1 element:

\[
\begin{align*}
T_{QS}^{\text{worst}}(n) &= T_{QS}^{\text{worst}}(n - 1) + T_{QS}^{\text{worst}}(1) + \Theta(n) \\
&= T_{QS}^{\text{worst}}(n - 1) + \Theta(n) = T_{QS}^{\text{worst}}(n - 2) + \Theta(n - 1) + \Theta(n) \\
&= \ldots = \Theta(1) + \ldots + \Theta(n - 1) + \Theta(n) \in \Theta(n^2)
\end{align*}
\]
Time Complexity of QuickSort – Special Cases?

What happens if:

- A is already sorted?

→ partition sizes always 1 and n-1
⇒ Θ(n^2)

- A is sorted in reverse order?
→ partition sizes always 1 and n-1
⇒ Θ(n^2)

- one partition has always at most a elements (for a fixed a)?
→ same complexity as a = 1
⇒ Θ(n^2)

- partition sizes are always n(1 - a) and na with 0 < a < 1?
→ same complexity as best case
⇒ Θ(n log n)

Questions:

- What happens in the “usual” case?
- Can we force the best case?
Time Complexity of QuickSort – Special Cases?

What happens if:

- A is already sorted?
  → partition sizes always 1 and n-1 ⇒ \( \Theta(n^2) \)
Time Complexity of QuickSort – Special Cases?

What happens if:

- A is already sorted?
  → partition sizes always 1 and n-1 ⇒ $\Theta(n^2)$
- A is sorted in reverse order?
- one partition has always at most $a$ elements (for a fixed $a$)?
  → same complexity as $a = 1$ ⇒ $\Theta(n^2)$
- partition sizes are always $n(1-a)$ and $na$ with $0 < a < 1$?
  → same complexity as best case ⇒ $\Theta(n \log n)$

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• one partition has always at most \( a \) elements (for a fixed \( a \))?
Time Complexity of QuickSort – Special Cases?

What happens if:

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  → partition sizes always 1 and n-1 ⇒ Θ(n²)
- A is sorted in reverse order?
  → partition sizes always 1 and n-1 ⇒ Θ(n²)
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Time Complexity of QuickSort – Special Cases?

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- partition sizes are always $n(1 - a)$ and $na$ with $0 < a < 1$?
Time Complexity of QuickSort – Special Cases?

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  → same complexity as a = 1 ⇒ Θ(n²)

- Partition sizes are always n(1 – a) and na with 0 < a < 1?
  → same complexity as best case ⇒ Θ(n log n)
Time Complexity of QuickSort – Special Cases?

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  → partition sizes always 1 and n-1 ⇒ $\Theta(n^2)$
- A is sorted in reverse order?
  → partition sizes always 1 and n-1 ⇒ $\Theta(n^2)$
- one partition has always at most $a$ elements (for a fixed $a$)?
  → same complexity as $a = 1 ⇒ \Theta(n^2)$
- partition sizes are always $n(1 - a)$ and $na$ with $0 < a < 1$?
  → same complexity as best case ⇒ $\Theta(n \log n)$

Questions:

- What happens in the “usual” case?
- Can we force the best case?
Randomized QuickSort

RandPartition ( A: Array [p..r] ) : Integer {
   // choose random integer i between p and r
   i := rand(p,r);
   // make A[i] the (new) Pivot element:
   exchange A[i] and A[p];
   // call Partition with new pivot element
   q := Partition (A);
   return q;
}

RandQuickSort ( A:Array [p..r] ) {
   if p >= r then return;
   q := RandPartition(A);
   RandQuickSort (A[p..q]);
   RandQuickSort (A[q+1 ..r]);
}
Time Complexity of RandQuickSort

Best/Worst-case complexity?
Time Complexity of RandQuickSort

Best/Worst-case complexity?

- RandQuickSort may still produce the worst (or best) partition in each step
- worst case: $\Theta(n^2)$
- best case: $\Theta(n \log n)$
Time Complexity of RandQuickSort

Best/Worst-case complexity?

- RandQuickSort may still produce the worst (or best) partition in each step
- worst case: $\Theta(n^2)$
- best case: $\Theta(n \log n)$

However:

- it is not determined which input sequence (sorted order, reverse order) will lead to worst case behavior (or best case behavior);
- any input sequence might lead to the worst case or the best case, depending on the random choice of pivot elements.

Thus: only the **average-case complexity** is of interest!
Average Case Complexity of RandQuickSort

Assumptions:

- we compute $\bar{T}_{RQS}(A)$, i.e., the expected run time of RandQuickSort for a given input $A$
- $\text{rand}(p, r)$ will return uniformly distributed random numbers (all pivot elements have the same probability)
- all elements of $A$ have different size: $A[i] \neq A[j]$. 


Average Case Complexity of RandQuickSort

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• $\text{rand}(p, r)$ will return uniformly distributed random numbers (all pivot elements have the same probability)
• all elements of $A$ have different size: $A[i] \neq A[j]$

Basic Idea:

• only count number of comparisons between elements of $A$
• let $z_i$ be the $i$-th smallest element in $A$
• define

$$X_{ij} = \begin{cases} 
1 & \text{if } z_i \text{ is compared to } z_j \\
0 & \text{otherwise}
\end{cases}$$

• random variable $T_{RQS}(A) = \sum_{i<j} X_{ij}$
Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$\bar{T}_{RQS}(A) = E \left[ \sum_{i<j} X_{ij} \right]$$
Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$\bar{T}_{RQS}(A) = E \left[ \sum_{i<j} X_{ij} \right]$$

$$= \sum_{i<j} E \left[ X_{ij} \right]$$
Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

\[ \bar{T}_{RQS}(A) = E \left[ \sum_{i<j} X_{ij} \right] \]

\[ = \sum_{i<j} E \left[ X_{ij} \right] \]

\[ = \sum_{i<j} \Pr \left[ z_i \text{ is compared to } z_j \right] \]
Average Case Complexity of RandQuickSort

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\[ = \sum_{i<j} E \left[ X_{ij} \right] \]

\[ = \sum_{i<j} \Pr [z_i \text{ is compared to } z_j] \]

- suppose an element between \( z_i \) and \( z_j \) is chosen as pivot \textbf{before} \( z_i \) or \( z_j \) are chosen as pivots; then \( z_i \) and \( z_j \) are never compared
Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

\[ \bar{T}_{RQS}(A) = E \left[ \sum_{i<j} X_{ij} \right] \]

\[ = \sum_{i<j} E \left[ X_{ij} \right] \]

\[ = \sum_{i<j} Pr \left[ z_i \text{ is compared to } z_j \right] \]

- suppose an element between \( z_i \) and \( z_j \) is chosen as pivot before \( z_i \) or \( z_j \) are chosen as pivots; then \( z_i \) and \( z_j \) are never compared
- if either \( z_i \) or \( z_j \) is chosen as the first pivot in the range \( z_i, \ldots, z_j \), then \( z_i \) will be compared to \( z_j \)
Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

\[
\bar{T}_{RQS}(A) = E \left[ \sum_{i<j} X_{ij} \right] = \sum_{i<j} E \left[ X_{ij} \right] = \sum_{i<j} Pr \left[ z_i \text{ is compared to } z_j \right]
\]

- suppose an element between \( z_i \) and \( z_j \) is chosen as pivot before \( z_i \) or \( z_j \) are chosen as pivots; then \( z_i \) and \( z_j \) are never compared
- if either \( z_i \) or \( z_j \) is chosen as the first pivot in the range \( z_i, \ldots, z_j \), then \( z_i \) will be compared to \( z_j \)
- this happens with probability

\[
\frac{2}{j - i + 1}
\]
Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$\bar{T}_{\text{RQS}}(A) = 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j - i + 1}$$
Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

\[ \bar{T}_{RQS}(A) = 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j - i + 1} \]

\[ = 2 \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{1}{k} \]
Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

$$\bar{T}_{RQS}(A) = 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j - i + 1}$$

$$= 2 \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{1}{k}$$

$$\leq 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k}$$
Average Case Complexity of RandQuickSort

Expected Number of Comparisons:

\[ \overline{T}_{\text{RQS}}(A) = 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j - i + 1} \]

\[ = 2 \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{1}{k} \]

\[ \leq 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} \]

\[ = 2nH_n \]
**Average Case Complexity of RandQuickSort**

**Expected Number of Comparisons:**

\[
T_{RQS}(A) = 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j - i + 1}
\]

\[
= 2 \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{1}{k}
\]

\[
\leq 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k}
\]

\[
= 2nH_n
\]

\[
= O(n \log n)
\]
Part III

Outlook: Optimality of Comparison Sorts
Are Mergesort and Quicksort optimal?

**Definition**

**Comparison sorts** are sorting algorithms that use only comparisons (i.e. tests as $\leq, =, >, \ldots$) to determine the relative order of the elements.

**Examples:**
- InsertSort, BubbleSort
- MergeSort, (Randomised) Quicksort

**Question:**
Is $T(n) \in \Theta(n \log n)$ the best we can get (in the worst/average case)?
Decision Trees

Definition

A **decision tree** is a binary tree in which each internal node is annotated by a comparison of two elements. The leaves of the decision tree are annotated by the respective permutations that will put an input sequence into sorted order.
Decision Trees – Properties

Each comparison sort can be represented by a decision tree:
- a path through the tree represents a sequence of comparisons
- sequence of comparisons depends on results of comparisons
- can be pretty complicated for Mergesort, Quicksort, . . .

A decision tree can be used as a comparison sort:
- if every possible permutation is annotated to at least one leaf of the tree!
- if (as a result) the decision tree has at least n! (distinct) leaves.
A Lower Complexity Bound for Comparison Sorts

- A binary tree of height $h$ ($h$ the length of the longest path) has at most $2^h$ leaves.
- To sort $n$ elements, the decision tree needs $n!$ leaves.

**Theorem**

Any decision tree that sorts $n$ elements has height $\Omega(n \log n)$.

**Proof:**

- $h$ comparisons in the worst case are equivalent to a decision tree of height $h$
- with $h$ comparisons, we can sort $n$ elements (at best), if

$$n! \leq 2^h \iff h \geq \log(n!) \in \Omega(n \log n)$$

- because:

$$h \geq \log(n!) \geq \log \left( n^{n/2} \right) = \frac{n}{2} \log n$$
Optimality of Mergesort and Quicksort

**Corollaries:**
- MergeSort is an optimal comparison sort in the worst/average case
- QuickSort is an optimal comparison sort in the average case

**Consequences and Alternatives:**
- comparison sorts can be faster than MergeSort, but only by a constant factor
- comparison sorts can not be asymptotically faster
- sorting algorithms might be faster, if they can exploit additional information on the size of elements
- examples: **BucketSort**, CountingSort, RadixSort
Part IV

Bucket Sort – Sorting Beyond “Comparison Only”
Bucket Sort

Basic Ideas and Assumptions:

• pre-sort numbers in buckets that contain all numbers within a certain interval
• hope (assume) that input elements are evenly distributed and thus uniformly distributed to buckets
• sort buckets and concatenate them

Requires “Buckets”:

• can hold arbitrary numbers of elements
• can insert elements efficiently: in $O(1)$ time
• can concatenate buckets efficiently: in $O(1)$ time
• remark: linked lists will do
Implementation of BucketSort

BucketSort (A: Array[1..n]) {
  Create Array B[0..n−1] of Buckets;
  // assume all Buckets B[i] are empty at first

  for i from 1 to n do {
    insert A[i] into Bucket B[\text{floor}(n * A[i])];
  }

  for i from 0 to n−1 do {
    sort Bucket B[i];
  }

  concatenate Buckets B[0], B[1], ..., B[n−1] into A
}
Number of Operations of BucketSort

Operations:

- $n$ operations to distribute $n$ elements to buckets
- plus effort to sort all buckets
Number of Operations of BucketSort

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- if each bucket gets 1 element, then $\Theta(n)$ operations are required
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- $n$ operations to distribute $n$ elements to buckets
- plus effort to sort all buckets

Best Case:
- if each bucket gets 1 element, then $\Theta(n)$ operations are required

Worst Case:
- if one bucket gets all elements, then $T(n)$ is determined by the sorting algorithm for the buckets
Bucketsort – Average Case Analysis

- probability that bucket \( i \) contains \( k \) elements:

\[
P(n_i = k) = \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k}
\]

- expected mean and variance for such a distribution:

\[
E[n_i] = n \cdot \frac{1}{n} = 1 \quad \text{Var}[n_i] = n \cdot \frac{1}{n} \left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)
\]

- InsertionSort for buckets \( \Rightarrow \leq cn^2 \in O(n_i^2) \) operations per bucket
- expected operations to sort one bucket:

\[
\bar{T}(n_i) \leq \sum_{k=0}^{n-1} P(n_i = k) \cdot ck^2 = cE[n_i^2]
\]
Bucket sort – Average Case Analysis (2)

- theorem from statistics:
  \[ E[X^2] = E[X]^2 + \text{Var}(X) \]

- expected operations to sort one bucket:
  \[ \bar{T}(n_i) \leq c E[n_i^2] = c \left( E[n_i]^2 + \text{Var}[n_i] \right) = c \left( 1^2 + 1 - \frac{1}{n} \right) \in \Theta(1) \]

- expected operations to sort all buckets:
  \[ \bar{T}(n) = \sum_{i=0}^{n-1} \bar{T}(n_i) \leq c \sum_{i=0}^{n-1} \left( 2 - \frac{1}{n} \right) \in \Theta(n) \]

  (note: expected value of the sum is the sum of expected values)