Sensor Based Robotic Manipulation and Locomotion

Exercise Block 1
27.11.2016

Exercise 1

Derive the tensor transformation of a tensor $A$ of type $(0,2)$ between two local coordinates $x$ and $y$, given that the vectors transform with $\dot{y} = J(x)\dot{x}$.

Exercise 2

Consider an actuator consisting of a motor with inertia $J_m = 0.04\text{kg}\cdot\text{m}^2$, a gearbox with gear ratio 5:1 (i.e. $\tau_l = 5\tau_m$) and link inertia $J_l = 1\text{kg}\cdot\text{m}^2$. Assume that the controller on motor side is just a PD controller with damping $d$ and spring stiffness $k_m = 4\text{Nm/ rad}$, i.e. $\tau_m = k_m(\theta_d - \theta)$ with $\theta$ being the motor coordinate, $\theta_d$ the desired equilibrium and $\tau_m$ being the motor torque. The link is in contact with an environment which has the stiffness $k_l = 100\text{Nm/ rad}$. Which will be the oscillation frequency of the system, if deviated from equilibrium in the absence of damping, i.e. $d = 0$? Find $d$ such that the system has two identical, real eigenvalues.

![PD-controlled joint in contact with elastic environment](image)

Figure 1: PD-controlled joint in contact with elastic environment

Exercise 3

The dynamics of a two joint robot with two collinear, translational joints in the horizontal plane (Fig. 1) is:

$$
\tau_1 = (m_1 + m_2)\ddot{q}_1 + m_2\ddot{q}_2 \\
\tau_2 = m_2(\ddot{q}_1 + \ddot{q}_2)
$$
Therein, $\tau_i$ are the forces, $m_i$ the masses and $\ddot{q}_i$ the accelerations of the joints. The position of the TCP is described by the coordinate $x$.

a) Write the Jacobian of the manipulator. What is the dimension of the task space and of the nullspace?

b) Derive the manipulator dynamics in Cartesian coordinates.

c) Write a simple passivity based controller in task coordinates for this system. Write the Lyapunov-function and determine its derivative.

**Exercise 4**

Suppose that the system from Fig. 1 is controlled using a PD controller in joint coordinates and the P-gains are $k_{p1}$ and $k_{p2}$. What is the equivalent stiffness of the manipulator at the TCP (in task coordinates)? Check the result with the tensor transformation and the Jacobian from 1.a). Which representation is more appropriate therefore: the stiffness or the compliance tensor?

**Exercise 5**

How do you compute the serial interconnection of two multidimensional springs using stiffness matrices $K_1$ and $K_2$? And using compliance matrices $C_1$ and $C_2$?

The stiffness matrix gives the relation between an infinitesimal displacement vector $\Delta x$ and the elastic force covector $f$, i.e. $f = K \Delta x$ while the compliance matrix is the inverse of the stiffness matrix, satisfying $\Delta x = Cf$.

**Exercise 6**

Given is the redundant manipulator with joint coordinates $q \in \mathbb{R}^n$ and task coordinates $x \in \mathbb{R}^m$ with $m < n$. A controller implements in joint coordinates the stiffness rule

$$\tau = -K \Delta q, \quad K \in \mathbb{R}^{nxn} \quad (1)$$

Determine the equivalent Cartesian stiffness $K_C$, defined as $f = -K_C \Delta x$, with $f$ being the force dual to $\dot{x}$

a) Derive the relation without using the pseudo inverse.

b) Derive the relation with the pseudo inverse. Which is the right weighting of the pseudo inverse in this case to obtain the correct result?
Exercise 7

Given are two manifolds $M$ and $N$ with the local coordinates $x$ and $y = \phi(x)$. How does a third order tensor of type $(3,0)$ with the elements $c^x_{ijk}$ on $M$ transform to a tensor $c^y_{lmn}$ on $N$? The tensor is defined through the tri-liner mapping $V = \sum_{ijk} c^x_{ijk} v^x_i v^x_j v^x_k$, where $V \in \mathbb{R}$ and $v^x$ are arbitrary tangent vectors in the tangent space of $M$.

Hint: derive first, how an element $v^y_i$ can be represented in terms of $v^x$. Use throughout the derivation the above mentioned summation notation instead of classical matrix and vector operations!!

Exercise 8

The joint with antagonistic actuators from Fig. 3 is actuated by two position controlled, linear motors with positions $\theta_1$ and $\theta_2$. The actuators have the stiffness $k_1$ and $k_2$, respectively. The link side angle is $q = -x_1/r = x_2/r$, with $r$ being the joint radius.

a) Determine the Jacobian $P$ defined by $\dot{x} = P\dot{q}$, $x = (x_1, x_2)^T$. Which is the relation between the force vector $f = (f_1, f_2)^T$ and the joint torque $\tau$?

b) What is the dimension of the task space and of the joint space, respectively? Derive the stiffness in task coordinates under the assumption that the motor positions are kept constant ($\theta_1 = \theta_2 = 0$).

c) Which condition must be fulfilled by the force vector $f$ in order to be in the nullspace of $P^T$? What is its physical interpretation?

Exercise 9

Consider the redundant serial manipulator structure from Fig. 4, with elastic elements connected in parallel to the joints. Such designs are typically used as a counterbalance for high weight payloads for weight supporting systems with small joint motion range. Let the Jacobian be denoted by $J(q)$ and the total joint stiffness be denoted by $K_m = \text{diag}\{k_i\}$, with $k_i$ being the individual joint stiffness values, $i = 1, \ldots, 4$. The dynamics of the robot is thus

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + K_m \Delta q = \tau.$$  

(2)
The equilibrium configuration is denoted by \( q_0 \). Assume that \( \tau \) is composed of a gravity compensation term in the equilibrium configuration and a nullspace component:

\[
\tau = g(q_0) + \tau_N.
\]  
(3)

Design a null-space projection matrix \( P(q) \) which ensures that an arbitrary joint torque \( \tau_0 \), projected by \( P(q) \) to produce \( \tau_N = P(q)\tau_0 \), would not only result in zero Cartesian force at the tip of the manipulator, but would also not generate any additional Cartesian displacement \( \Delta x \) in the static case (\( \dot{q} = 0, \ddot{q} = 0 \)). Note that the Cartesian displacement would result in general from a joint displacement \( \Delta q \) produced by the action of the torque \( \tau_N \) on the joint springs. Which weighting matrix needs to be chosen for the pseudoinverse for the computation of the projection matrix? Hint: for the small displacements caused by the spring deflections, approximate \( g(q) \) by its first order Taylor expansion around \( q_0 \).

Exercise 10

Consider a three-dimensional, translational manipulator. Write a simple, passivity based controller which controls the robot such that it is constrained to the upper hemisphere of a sphere of radius \( r_0 \). On the surface of the sphere it should not be constrained by controller forces, meaning that it should be freely externally movable (for example by the user’s hand).

a) Define appropriate task coordinates.

b) Formulate the controller in these coordinates.

c) Derive the translational forces exerted on the tip of the robot by the controller.

Exercise 11

Consider a three-dimensional, translational manipulator. Write a simple, passivity based controller which controls the robot such that it is constrained to the surface of a endless cylinder of radius \( r_0 \). On the surface of the cylinder it should not be constrained by controller forces, meaning that it should be freely
externally movable (for example by the users’s hand).

a) Define appropriate task coordinates.
b) Formulate the controller in these coordinates.
c) Derive the translational forces exerted on the tip of the robot by the controller.

Exercise 12

Consider two segments of a two arm robot, as sketched in Fig. 5, with \((b < 2a)\).

a) Compute the distance \(d\) between \(A\) and \(B\) and define a repelling potential which prevents the collision of the two segment tips \(A\) and \(B\).
b) Compute the Jacobian and the resulting torques for the two joints.

Figure 5: Two repelling robot links